

HYPERBOLIC GRAPHS OF SURFACE GROUPS

Abstract

We give a sufficient condition under which the fundamental group of a reglued graph of surfaces is hyperbolic. A reglued graph of surfaces is constructed by cutting a fixed graph of surfaces along the edge surfaces, then regluing by pseudo-Anosov homeomorphisms of the edge surfaces. By carefully choosing the regluing homeomorphism, we construct an example of such a reglued graph of surfaces, whose fundamental group is not abstractly commensurate to any surface-by-free group, i.e., which is different from all the examples given in the paper [Mos97].

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1 Introduction

The fundamental group of the mapping torus of a pseudo-Anosov homeomorphism of an oriented closed hyperbolic surface is hyperbolic. This was first proved by Thurston. A direct proof was given by Bestvina and Feighn [BF92]. Using their idea, Mosher [Mos97] proved the following theorem.

Consider an oriented closed hyperbolic surface S . Let $\Phi_1, \dots, \Phi_m \in MCG(S)$ be an independent set of pseudo-Anosov mapping classes of S , and let $\phi_1, \dots, \phi_m \in Homeo(S)$ be pseudo-Anosov representatives of Φ_1, \dots, Φ_m respectively. If i_1, \dots, i_m are large enough positive integers, then the fundamental group of the graph of spaces \mathcal{G} , as shown in Figure 1, is a hyperbolic group. In the statement of this theorem, by

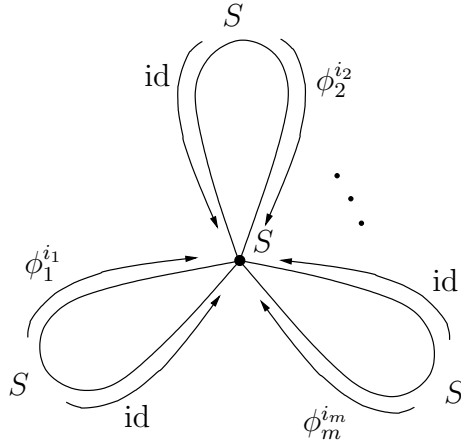


Figure 1:

saying a set B of pseudo-Anosov mapping classes is *independent*, we mean the sets $Fix(\Phi)$ are pairwise disjoint for $\Phi \in B$, where $Fix(\Phi)$ consists of the attractor and the repeller of Φ on the space of projective measured foliations $\mathcal{PMF}(S)$.

A *graph of surfaces* $S\Gamma$ consists of an oriented connected finite underlying graph Γ , a function which assigns to each vertex a closed hyperbolic surface or orbifold, to each edge a closed hyperbolic surface, and another function which assigns to each oriented edge a covering map from the edge surface to the vertex surface of the origin of the edge. In the cases studied in this paper, we change the canonical graph of surfaces by cutting along the edge surfaces, then choosing pseudo-Anosov homeomorphisms of the edge surfaces, then regluing. We call it a *graph of surfaces with pseudo-Anosov regluing*. Thus the mapping torus of a pseudo-Anosov homeomorphism can be considered as this type of space whose underlying graph consists of only one vertex and one edge, and the vertex and edge spaces are the same hyperbolic surface. The case studied by Mosher is another reglued graph of surfaces with the underlying graph consists of only one vertex, in addition the vertex and edge spaces are the same hyperbolic surface S .

We shall extend Mosher's theorem to the general graphs of surfaces with pseudo-Anosov regluing. Theorem 1 says that if the pseudo-Anosov homeomorphisms are

chosen to satisfy an appropriate independence condition, then the fundamental group of the reglued graph of surfaces is word hyperbolic, when these homeomorphisms are replaced with sufficiently high powers of themselves.

We shall describe this cutting and regluing process with more details. Let ST be a graph of surfaces with the underlying graph Γ , let E be the set of oriented edges of Γ , and let V be the set of vertices of Γ . For each $e \in E$, let S_e be the corresponding edge surface. For each oriented edge e , there is a finite covering map $p_e : S_e \rightarrow F_{o(e)}$, where $F_{o(e)}$ is the vertex surface of the origin $o(e)$ of the edge e . For each inverse pair of oriented edges e, \bar{e} , there is an inverse pair of homeomorphisms $g_e : S_e \rightarrow S_{\bar{e}}, g_e^{-1} : S_{\bar{e}} \rightarrow S_e$. Let $\varphi = \{\phi_e \mid e \in E\}$, where $\phi_e : S_e \rightarrow S_e$ is a pseudo-Anosov homeomorphism of S_e . Let ST_φ be the graph of surfaces with pseudo-Anosov regluing obtained from ST by cutting along each S_e and regluing using ϕ_e , i.e., in the reglued graph of surfaces, the effect is to replace the map $g_e : S_e \rightarrow S_{\bar{e}}$ by the map $g_e \circ \phi_e$, for $e \in E$. Moreover, let $\mathbf{m} = \{m_e \mid e \in E\}$, where m_e are positive integers, and let $ST_{\varphi^{\mathbf{m}}}$ be the graph of surfaces obtained from ST by regluing using $\phi_e^{m_e}$ for each $e \in E$.

Given a vertex v of the underlying graph Γ , let F_v be the corresponding vertex surface (or orbifold). For each $v \in V$, denote $I_v = \{i \mid e_i \text{ is an oriented edge such that the origin of } e_i \text{ is } v\}$. For each $v \in V$ and each $i \in I_v$, there is a finite index covering map $p_i : S_i \rightarrow F_v$, where S_i is a shorthand notation of S_{e_i} . For an oriented edge e_i has the vertex v as both of its origin and terminal, the covering maps $S_i \xrightarrow{p_i} F_v$ and $S_i \xrightarrow{g_i} S_{\bar{i}} \xrightarrow{p_{\bar{i}}} F_v$ might be different, where g_i is a shorthand notation of g_{e_i} . The portion of $ST_{\varphi^{\mathbf{m}}}$ around a vertex space F_v could look like in Figure 2

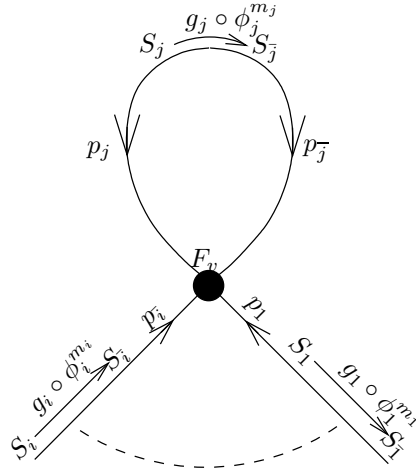


Figure 2:

For the purpose of Theorem 1, fix a hyperbolic structure on each vertex surface F_v . For each $v \in V$ and each $i \in I_v$, suppose S_i equipped with the pullback metric by the covering map $p_i : S_i \rightarrow F_v$. Hence for each covering map p_i , there is the derivative map $Dp_i : PS_i \rightarrow PF_v$, where PS_i, PF_v are the projective tangent bundles of S_i and

F_v respectively. For an oriented edge e_j , let $\phi_j^{m_j} : S_j \rightarrow S_j$ be the pseudo-Anosov homeomorphism for the edge e_j , with the stable geodesic lamination $\Lambda_j^s \subset S_j$; the stable geodesic lamination $\Lambda_j^s \subset S_j$ of $(\phi_j^{m_j}) = g_j \phi_j^{-m_j} g_j^{-1}$ is homotopic to the image under g_j of the unstable geodesic lamination of $\phi_j^{m_j}$. The geodesic laminations Λ_j^s and Λ_j^u are independent of the choice of the exponent m_j . In the following, let $T\Lambda_i^s$ denote the unit tangent vector space of Λ_i^s .

The main theorem of this paper is

Theorem 1. *Let ST_{φ^m} be a graph of surfaces with pseudo-Anosov regluing. Let Γ be its underlying graph. If for each vertex $v \in \Gamma$, and for each $i \in I_v$, the derivative maps $Dp_i|T\Lambda_i^s$ are injections, and their images are disjoint compact subsets of PF_v , then the fundamental group of ST_{φ^m} is hyperbolic, when $m_i \in \mathbf{m}$ are sufficiently large.*

The proof of the hyperbolicity of ST_{φ^m} depends ultimately on the Combination Theorem of [BF92]. The Combination Theorem says that if the quasi-isometrically embedded condition (which is automatically satisfied in the cases studied in this paper) and the hallways flare condition (which is much more difficult to check) both hold, then ST_{φ^m} is a hyperbolic space. In order to check the satisfaction of the hallways flare condition, we need to extend the parallel corresponds lemma [Mos97], the key in that paper, to a new version of the parallel corresponds lemma.

The idea of the proof of Theorem 1 is: by applying the new version of parallel corresponds lemma, if the hypothesis of Theorem 1 is satisfied, then the hallways flare condition is satisfied. Therefore the fundamental group of ST_{φ^m} is hyperbolic.

Here are some applications of this theorem.

First: let S be a closed hyperbolic surface, let G, H be finite subgroups of the mapping class group $MCG(S)$, and let $\Phi \in MCG(S)$ be a pseudo-Anosov mapping class. Suppose G, H each have trivial intersection with the virtual centralizer of $\langle \Phi \rangle$ in $MCG(S)$, then for sufficiently large n , the subgroup A of $MCG(S)$ generated by $G, \Phi^n H \Phi^{-n}$ is isomorphic to the free product of these subgroups. Even more, A is a virtual Schottky subgroup of $MCG(S)$, in the sense of [FM02a].

Second: let \mathcal{G}_{ϕ^m} be a graph of surfaces with regluing as in Figure 3, where S, F

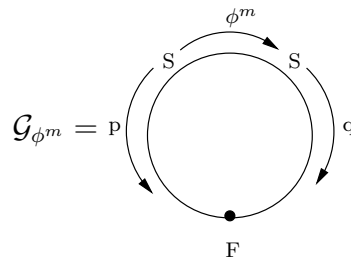


Figure 3:

are genus 3 and 2 tori, $\phi : S \rightarrow S$ is a pseudo-Anosov homeomorphism. Suppose there exist simple closed curves $a \subset F$ and $c \subset S$, as shown in Figure 4, such that

$p^{-1}(a) = c$, $c \subset q^{-1}(a)$, and $q^{-1}(a)$ is disconnected. In addition, suppose that in the group $MCG(S)$, the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with the deck transformation groups of p and q , where Φ is the mapping class of ϕ . Then $\pi_1(\mathcal{G}_{\phi^m})$ is hyperbolic when m is sufficiently large.

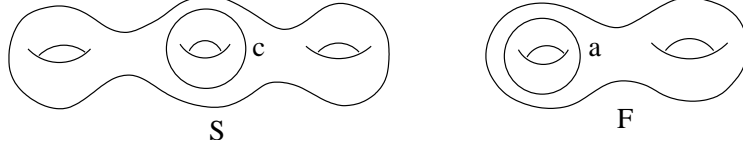


Figure 4:

More interesting, we will see that there exists a pseudo-Anosov homeomorphism ϕ of S , such that $\pi_1(\mathcal{G}_{\phi^m})$ is not commensurate to $\pi_1(S') \rtimes K$, for any oriented, closed hyperbolic surface S' , and for any free group K , where \mathcal{G}_{ϕ^m} as the above. More than that, $\pi_1(\mathcal{G}_{\phi^m})$ is not even quasi-isometric to any surface-by-free group. Therefore $\pi_1(\mathcal{G}_{\phi^m})$ is different from all the hyperbolic groups constructed in [Mos97].

Problems. Do there exist some reducible homeomorphisms of the edge surfaces, such that the graph of surfaces with reducible homeomorphism regluing are hyperbolic?

Is Theorem 1 still true when the vertex and edge groups are free groups?

2 Preliminaries

In this section, we recall some preliminaries about combinatorial and geometric group theory, and some facts of hyperbolic geometry which will be used later.

Graphs of surfaces (The material in this subsection can be found in [SW79] and [Ser80])

Let Γ be a connected finite graph, let e be an oriented edge of Γ , and let \bar{e} be the inverse edge of e . The vertex $o(e)$ is called the origin of e and the vertex $t(e)$ is called the terminal of e , obviously $o(e) = t(\bar{e})$.

A *graph of surfaces* $S\Gamma$ consists of a connected finite graph Γ and a function which assigns to each vertex $v \in \Gamma$ a closed hyperbolic surface or orbifold F_v , to each pair of oriented edges e, \bar{e} closed hyperbolic surfaces $S_e, S_{\bar{e}}$ and an inverse pair of homeomorphisms $S_e \rightarrow S_{\bar{e}}, S_{\bar{e}} \rightarrow S_e$, and to each edge e a continuous map $p_e : S_e \rightarrow F_{o(e)}$, such that p_e induces an injection on the fundamental groups. In most of our cases p_e are covering maps for every edge e of Γ .

Given a graph of surfaces $S\Gamma$, we can define the *total space* S_Γ as the quotient of the disjoint union $(\cup\{F_v|v \in V(\Gamma)\}) \cup (\cup\{S_e \times I|e \in E(\Gamma)\})$ by identifying the equivalent classes: $(s, 0) \sim p_e(s)$ for $(s, 0) \in S_e \times 0, p_e(s) \in F_{o(e)}$; $(s, 1) \sim p_{\bar{e}}(s)$ for

$(s, 1) \in S_e \times 1$, $p_{\bar{e}}(s) \in F_{t(e)}$. The *fundamental group of the graph of surfaces* $\pi_1(ST)$ is defined to be the fundamental group of the total space S_Γ . There is a projection map $\pi : S_\Gamma \rightarrow \Gamma$ such that each vertex surface F_v maps to the vertex v and each $S_e \times I$ maps to the edge e , π is an onto map.

The universal cover \widetilde{ST} of ST is a union of copies of the universal covers $\widetilde{S}_e \times I$ and \widetilde{F}_v . In \widetilde{ST} , if we identify each copy of \widetilde{F}_v to a point and each copy of $\widetilde{S}_e \times I$ to a copy of I , then we obtain a graph t and there is a canonical projection map $\widetilde{\pi} : \widetilde{ST} \rightarrow t$. It is not hard to see that t is a tree, called the *Bass-Serre tree*. The action of $\pi_1(ST)$ on \widetilde{ST} descends to an action of $\pi_1(ST)$ on t , where the quotient graph coincides with the original graph Γ , and the stabilizers of each vertex and each edge of t are conjugates of corresponding fundamental groups of F_v and S_e .

The Bestvina – Feighn Combination Theorem (The material in this subsection can be found in [BF92])

For the purpose of this paper, instead of the original combination theorem, we shall state the tailored Bestvina-Feighn Combination Theorem in the context of the graphs of surfaces only.

Let ST be a graph of surfaces with the underlying graph Γ , and let $\pi : ST \rightarrow \Gamma$ be the projection map. Denote the preimage of the midpoint of an edge $e \in \Gamma$ under p by S_e . For a vertex $v \in \Gamma$, we consider the component containing v of Γ cut open along the midpoints of edges. Let X_v denote the preimage of this component under p , called *vertex space*. For any vertex $v \in \Gamma$, the vertex surface F_v is a deformation retract of the vertex space X_v .

For each edge e of Γ , the lift to the universal covers of the finite index covering map $p_e : S_e \rightarrow F_{o(e)}$ is a quasi-isometry. This is precisely the 'quasi-isometrically embedded condition' in [BF92]. We may omit this condition from the hypothesis of the combination theorem for the cases of the graphs of surfaces.

Define a continuous function $\Delta : [-k, k] \times I \rightarrow \widetilde{ST}$ to be a *hallway* of length $2k$, if for any i from $-k$ to k , $\Delta(\{i\} \times I)$ is a geodesic in $\widetilde{S}_{e(i)}$, and $\Delta((i, i+1) \times I)$ stays in the interior of $\widetilde{X}_{v(i)}$. Suppose $\Delta([i, i+1] \times I)$ stays in the closure of $\widetilde{X}_{v(i)}$. Δ is ρ -thin if $d_{\widetilde{X}_{v(i)}}(\Delta((i, t)), \Delta((i+1, t))) \leq \rho$ for $i \in \{-k, -k+1, \dots, k-1\}$ and $t \in I$. The hallway Δ is *essential* if the edge path $e(-k) * \dots * e(k)$ never backtracks in the Bass-Serre tree t , i.e., $e(i) \neq \bar{e}(i+1)$ for $i \in \{-k, \dots, k-1\}$. The *girth* of Δ is the length of $\Delta(\{0\} \times I)$. Let $\lambda > 1$. The hallway Δ is λ -hyperbolic if $\lambda l(\Delta(\{0\} \times I)) \leq \max\{l(\Delta(\{-k\} \times I)), l(\Delta(\{k\} \times I))\}$. The graph of surfaces ST is said to satisfy the *hallways flare condition* if there exist numbers $\lambda > 1$ and $k \geq 1$ such that for any ρ there exists a constant $H(\rho)$, such that any ρ -thin essential hallway of length $2k$ and girth at least $H(\rho)$ is λ -hyperbolic.

Theorem 2. (*Combination Theorem*) *Let ST be a graph of surfaces. Suppose that ST satisfies the hallways flare condition, then the fundamental group of ST is hyperbolic.*

Remarks: 1. Notice that in the Bestvina-Feighn's combination theorem, the vertex spaces are used in defining the hallways; but in the proof of Theorem 1, we use the vertex surfaces instead. We are allowed to do so, because the vertex surface is a

retraction of the corresponding vertex space, and their universal covers are quasi-isometric to each other.

2. The hallways in the combination theorem are 'edge hallways', i.e., the rungs $\Delta(i) \times I$ of the hallway Δ are geodesics in the edge surfaces. In Theorem 1, the hallways are 'vertex hallways', i.e., $\Delta(i) \times I$ are geodesics in the vertex surfaces. Since the covering map from an edge surface to a vertex space is a quasi-isometry, and the vertex spaces and the vertex surfaces are quasi-isometric, if the vertex hallways flare condition is satisfied then the edge hallways flare condition is satisfied.

3. For the cases studied in this paper, we will prove the hallways flares condition for length 2 hallways only.

Construction of pseudo-Anosov homeomorphisms (The materials is covered by [Pen88])

In a surface S , \mathcal{C} is an *essential curve system*, if $\mathcal{C} = \{c_1, \dots, c_n\}$, where c_1, \dots, c_n are non-trivial simple closed curves on S which are pairwise disjoint and pairwise non-homotopy.

Let \mathcal{C} and \mathcal{D} be two disjoint essential curve systems, \mathcal{C} hits \mathcal{D} *efficiently* if \mathcal{C} intersect \mathcal{D} transversely, and no component on $S \setminus (\mathcal{C} \cup \mathcal{D})$ is a *bigon*, an interior of a disc whose boundary consists of one arc of $C \in \mathcal{C}$ and one arc of $D \in \mathcal{D}$. We say that $\mathcal{C} \cup \mathcal{D}$ *fills* S if the components of the complement of $(\mathcal{C} \cup \mathcal{D})$ are disks.

The following shows how to construct pseudo-Anosov homeomorphisms.

Theorem 3. ([Pen88]) *Suppose that \mathcal{C} and \mathcal{D} are essential curve systems in an oriented surface F so that \mathcal{C} hits \mathcal{D} efficiently and $\mathcal{C} \cup \mathcal{D}$ fills F . Let $R(\mathcal{C}^+, \mathcal{D}^-)$ be the free semigroup generated by the Dehn twists $\{\tau_c^+ : c \in \mathcal{C}\} \cup \{\tau_d^{-1} : d \in \mathcal{D}\}$. Each component map of the isotopy class of $\omega \in R(\mathcal{C}^+, \mathcal{D}^-)$ is either the identity or pseudo-Anosov, and the isotopy class of ω is itself pseudo-Anosov if each τ_c^+ and τ_d^{-1} occur at least once in ω .*

Surface group extensions (The materials is covered by [Mos] and [FM02c])

A *surface group extension* is a short exact sequence of the form

$$1 \rightarrow \pi_1(S, x) \rightarrow \Gamma \rightarrow G \rightarrow 1 \quad (1)$$

where S is a closed, oriented surface of genus $g \geq 2$. The canonical example is the sequence

$$1 \rightarrow \pi_1(S, x) \xrightarrow{i} MCG(S, x) \xrightarrow{q} MCG(S) \rightarrow 1 \quad (2)$$

where $MCG(S)$ is the mapping class group of S , $MCG(S, x)$ is the mapping class group of S punctured at x . This short exact sequence is universal for surface group extension, meaning that for any extension as in (1), there exists a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(S, x) & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \alpha \downarrow \\ 1 & \longrightarrow & \pi_1(S, x) & \xrightarrow{i} & MCG(S, x) & \xrightarrow{q} & MCG(S) \longrightarrow 1 \end{array} \quad (3)$$

where Γ is identified with the pushout group

$$\Gamma_\alpha = \{(\phi, \gamma) \in MCG(S, x) \times G \mid q(\phi) = \alpha(\gamma)\}, \quad (4)$$

α is a homomorphism from G to $MCG(S)$, and the homomorphisms $\Gamma \rightarrow G$ and $\Gamma \rightarrow MCG(S, x)$ are the projection homomorphisms of the pushout group. We are more interested in the case where α is an inclusion.

Virtual centralizer of Φ (The material is covered by [Mos])

Given a subgroup H of a group G , the *virtual centralizer* $VC(H)$ of H in G is the subgroup of all $g \in G$ which commute with a finite index subgroup of H . The virtual centralizer of an infinite cyclic pseudo-Anosov subgroup has a nice geometric description. Let $\mathcal{PML}(S)$ denote the space of projective measured laminations of the surface S . Let $\Lambda^s, \Lambda^u \subset \mathcal{PML}$ be the fixed points of a pseudo-Anosov mapping class Φ , and let $Fix\{\Lambda^s, \Lambda^u\}$ denote the subgroup in $MCG(S)$ whose elements fix Λ^s, Λ^u point wise. [Mos] shows that $Fix\{\Lambda^s, \Lambda^u\} = VC\langle\Phi\rangle$.

Facts of hyperbolic geometry (The material is covered by [BH99] and [CB])

Our proofs make heavy use of the following facts of hyperbolic space, H^2 , geometry:

Fact 1. For any $0 < \delta < 1$, and $D > 0$, there exists $l(\delta, D)$, such that if γ, α are geodesic segments of length at least $l(\delta, D)$, and the end points x, y of γ have distance at most D from the end points x', y' of α respectively, then there exist subsegments $\gamma' \subset \gamma, \alpha' \subset \alpha$ of lengths at least $(1 - \delta)Length(\gamma)$ and $(1 - \delta)Length(\alpha)$ respectively, such that the Hausdorff distance between γ' and α' is less than δ .

Roughly speaking, for any two geodesic segments, if their end points have bounded distances from each other, then most part of them can be arbitrarily close to each other as long as the segments are long enough.

Fact 2. Given $k \geq 1, c \geq 0$, there exists a constant $N_0(k, c)$, such that any (k, c) quasi-geodesics line or segment in H^2 has Hausdorff distance at most $N_0(k, c)$ from a geodesic line or segment with the same end points.

Fact 3. Let Λ_1 and Λ_2 be two minimal geodesic laminations filling a hyperbolic surface S . If their lifts $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ on the universal cover of \tilde{S} have at least one end point in common, then $\Lambda_1 = \Lambda_2$. A geodesic lamination Λ is *minimal* if every leaf L is dense, that is, $\overline{L} = \Lambda$. A geodesic lamination $\Lambda \subset S$ is a *filling* lamination if no simple closed curve in S is disjoint from Λ .

The reason is that two minimal filling surface geodesic laminations either transversely intersect with each other or are equal to each other.

From [FLP⁺79], we know that the stable and unstable geodesic laminations of a pseudo-Anosov homeomorphism are minimal and filling.

3 Main Theorem

We will give a new version of Mosher's parallel corresponds lemma and use it to prove Theorem 1. Moreover we will reformulate the hypothesis of Theorem 1. The original corresponds lemma of Mosher is in [Mos97].

3.1 New version of the parallel corresponds lemma

Consider a pseudo-Anosov mapping class $\Phi \in MCG(S)$, let $\phi \in Homeo(S)$ be a pseudo-Anosov representative with the stable and unstable measured foliations f_ϕ^s, f_ϕ^u . Recall that the transverse measures on f_ϕ^s and f_ϕ^u define a singular Euclidean structure on S , with isolated cone singularities. We call the leaves of f_ϕ^s the horizontal leaves and the leaves of f_ϕ^u the vertical leaves. The singular Euclidean structure determines a metric d_ϕ on S for which each path can be homotopic to a unique geodesic rel end points. The lifts to the universal covers of the hyperbolic metric and the singular Euclidean metric are quasi-isometric equivalent.

In the following, for a homotopy class γ of a curve rel. end points, let γ^h denote the hyperbolic geodesic segment in the homotopy class of γ , and let γ^E denote the singular Euclidean geodesic segment in the same homotopy class. Let $|\cdot|$ denote the hyperbolic metric, and let $|\cdot|_E$ denote the singular Euclidean metric. For a homotopy class γ , let $|\gamma|$ denote the hyperbolic length of γ^h , let $|\gamma|_E$ denotes the singular Euclidean length of γ^E .

Given $0 < \eta < 1$, define $slope_\phi^\eta$ to be the set of all homotopy classes γ , such that the (unsigned) Euclidean angle between γ^E and f_ϕ^s is at least η , on a subset of γ^E of length at least $\eta \cdot Length \gamma^E$. Given $\lambda > 1$, let $stretch_\phi^\lambda = \{\gamma \mid |(\phi(\gamma))| > \lambda|\gamma|\}$. Let n be a large enough integer, such that if the vector $v \in E^2$ has an angle at least η with the horizontal axis, then the matrix

$$\begin{pmatrix} \lambda_\phi^{-n} & 0 \\ 0 & \lambda_\phi^n \end{pmatrix}$$

stretches v by a factor of at least λ/η , where $\lambda_\phi = \lim_{i \rightarrow \infty} |\phi^i(\alpha)|^{1/i}$ is the stretching factor of ϕ , α is a simple closed geodesic on S . Since the singular Euclidean metric is quasi-isometric to the hyperbolic metric, it follows that given ϕ , $0 < \eta < 1$, and $\lambda > 1$, there exists N such that if $n \geq N$, then $slope_\phi^\eta \subset stretch_{\phi^n}^\lambda$.

An η -lever is a homotopy from a singular Euclidean geodesic segment α to a horizontal segment β , where β is a segment of a nonsingular leaf of the horizontal foliation f_ϕ^s , such that each track of the homotopy is a vertical geodesic segment, maybe degenerate, and each point of $\text{int}(\alpha)$ is disjoint from singularities during the homotopy, and $\text{int}(\alpha)$ makes an angle at most η with the horizontal leaves. In [Mos97], β is not necessary to be a segment of a nonsingular leaf. But we can always make β be a segment of a nonsingular leaf, because there exist nonsingular leafs which are arbitrary close to a singular leaf. Notice that the angle between a singular Euclidean geodesic and the horizontal leaves changes only when the singular Euclidean geodesic passes a singularity. Therefore the interior of α has a constant angle with the horizontal leaf.

A lever is denoted by (α, β) , where α is called the *inclined edge* of the lever, and β is called the *horizontal edge* of the lever. A lever is *maximal* if and only if a singularity contained in the track of each end point of α . The *length* of the lever is $|\alpha|_E$, the *height* of the lever is the maximum length of the tracks of the points of α , which is achieved at the endpoints.

Proposition 4. *For any $l, H > 0$, there exists $\eta(l, H) > 0$, so that every maximal η -lever has length at least l and height at most H .*

The proof is given in the first seven paragraphs of the proof of the sublemma on page 3451 in [Mos97]. This proposition will be used in the proof of the following lemma.

In the proof of the following lemma, we need some facts. It is well known that the measured foliations f_ϕ^s, f_ϕ^u can be straightened to measured geodesic laminations l_ϕ^s, l_ϕ^u . Actually, there is a 1-1 correspondence between leaves of l_ϕ^s and smooth leaves of f_ϕ^s , where a smooth leaf is either a nonsingular leaf or the union of two singular half-leaves meeting at a singularity with angle 180° . Similarly for f_ϕ^u . The singularities are discrete, so the length of any geodesic between them has a positive lower bound.

Lemma 5. *(New version of Parallel Corresponds lemma) Given any pseudo-Anosov homeomorphism ϕ and $0 < \epsilon < 1$, there exist $0 < \eta < 1$ and $L > 0$ such that for any homotopy class γ , if $\gamma \notin \text{slope}_\phi^\eta$ and $|\gamma|_E \geq L$, then on a subset of γ^h of length at least $(1 - \epsilon)\text{Length}(\gamma^h)$, the distance between the tangent line of γ^h and the set l_ϕ^s , measured in PS , is at most ϵ .*

The differences between the Parallel Corresponds lemma in [Mos97] and this new version are as follows. In [Mos97], the Parallel corresponds lemma works only for closed based geodesics, and the word metric is used to define the stretching factor; in this paper, the new version of the parallel corresponds lemma works for non closed geodesics as well, and the hyperbolic metric is used to define the stretching factor.

Proof. The first step is to find long subsegments $\alpha_i \subset \gamma^E$ and segments β_i of leaves of f_ϕ^s , such that α_i is homotopic to β_i by homotoping through short paths. Then we shall project α_i to a subsegment of γ^h and project β_i to a segment of a leaf B_i^h of l_ϕ^s , and show that a big portion of these projections are very close to each other. Finally we shall prove most part of γ^h are covered by big portion of these projections.

For $\gamma \notin \text{Slope}_\phi^\eta$, let $\{(\alpha_i, \beta_i)\}$ be the set of all maximal η -levers of γ^E , where the inclined edge α_i is a subsegment of γ^E and the horizontal edge β_i is a segment of some non-singular leaf B_i^E of f_ϕ^s .

Step 1: first, let $H = 1$, by proposition 4, we know that for any $l > 0$, there exists $\eta > 0$ such that every maximal η -lever $\{(\alpha_i, \beta_i)\}$ has length at least l and height at most $H = 1$. The first step is proven.

Step 2: we shall construct long subsegments of γ^h from the inclined edges $\alpha_i \subset \gamma^E$ of the maximal levers, such that these long subsegments of γ^h have small distance with l_ϕ^s measured in PS . In the rest of this lemma, the distance and length mean hyperbolic distance and length, otherwise we will use the notations Euclidean distance and length. I will use the notation "E" to represent the Euclidean distance and length.

We know that any non-singular leaf B_i^E of f_ϕ^s is a k, c quasi-geodesic under the hyperbolic metric, and it can be straightened to a unique leaf B_i^h of l_ϕ^s . Let $\delta_i \subset \gamma^h$ and $\sigma_i \subset B_i^h$ denote the closest point projections from $\alpha_i \subset \gamma^E$ to γ^h , and from $\beta_i \subset B_i^E$ to B_i^h respectively. We shall see that most portion of δ_i has small distance with σ_i , for all i .

Since γ^E is a k, c quasi-geodesic segment contained in the $N_0(k, c)$ neighborhood of γ^h , and δ_i, α_i are subsegments of γ^h, γ^E respectively, it follows that the distances

between the end points of δ_i and α_i are not greater than N_0 . For the same reason, the distance between the end points of σ_i and β_i are not greater than N_0 . The singular Euclidean distances between the end points of β_i and α_i is less than the height $H = 1$. The hyperbolic distances between the end point of them are at most mk for some $m > 0$, because the singular Euclidean and hyperbolic metric are k, c quasi-isometric to each other. Therefore the distances between the end points of δ_i and σ_i are less than $2N_0 + mk$. According to Fact 1, for any $\epsilon_1 > 0$, there exists L_1 depending on $2N_0 + mk$ and ϵ_1 , if the length of δ_i is greater than L_1 , then more than $(1 - \epsilon_1)|\delta_i|$ part of δ_i has distance less than ϵ_1 with σ_i .

The condition on the length of δ_i greater than L_1 is easy to satisfy. Since α_i is a quasi-geodesic segment whose end points have distances less than N_0 with the end points of δ_i , there exists $l_1 > 0$, such that if the Euclidean length of α_i is greater than l_1 , then the length of δ_i is greater than L_1 . By applying the step 1, we may now choose η small enough, so that the Euclidean length of α_i is greater than l_1 for any i . Therefore more than $(1 - \epsilon_1)|\delta_i|$ part of δ_i has distance less than ϵ_1 with σ_i .

So far, we have proved that for any ϵ_1 , there exists η , such that if $\gamma \notin \text{slope}_\phi^\eta$, then we can locate long subsegments δ_i of γ^h , such that more than $(1 - \epsilon_1)$ of the length of δ_i has distance less than ϵ_1 with $\sigma_i \subset B_i^h$, for any i .

Step 3: we will prove that $(1 - \epsilon_1) \sum_i |\delta_i|$ part of $\cup_i (\delta_i)$ covers most part of γ^h . We call this $(1 - \epsilon_1) \sum_i |\delta_i|$ part of $\cup_i (\delta_i)$ the 'good' part of γ^h .

Since $\gamma \notin \text{slope}_\phi^\eta$, on a subset of γ^E of length at least $(1 - \eta)|\gamma|_E$, the angle between γ^E and f_ϕ^s is less than η , i.e., the η -levers cover more than $(1 - \eta)$ part of γ^E . The worst situation is that the two end subsegments of γ^E are covered by η -levers with lengths less than l_1 . In this case, after straightening, the end subsegments of γ^h may not have distances less than ϵ_1 with B^h . We will only prove this lemma for the worst situation, i.e., more than $(1 - \epsilon_1)|\gamma|_E$ part of γ^E is covered by the union of the maximal η -levers (α_i, β_i) and two end η -levers which cover the two end segments of γ^E respectively and with lengths less than l_1 .

In the following the quasi-isometries will be replaced by bi-Lipschitz maps when dealing with long segments. In the rest of this proof, let $|\alpha_i|$ denote the length of the hyperbolic geodesic which is homotopic to α_i rel. end points, and let $|\alpha_i|_E$ denote the Euclidean length of the singular Euclidean geodesic α_i . Keep in mind that none of the following δ_i is the projection of an end subsegment of γ^E .

$$\begin{aligned} (1 - \epsilon_1) \sum_i |\delta_i| &\geq (1 - \epsilon_1) \left(\sum_i (|\alpha_i| - 2N_0) \right) \\ &\geq (1 - \epsilon_1) \left(\sum_i \left(\frac{|\alpha_i|_E}{k} - 2N_0 \right) \right) \end{aligned}$$

According to Proposition 4, we can take η to be small enough, so that $|\alpha_i|_E \geq l_2 = 4kN_0$ for any i

$$\geq (1 - \epsilon_1) \frac{\sum_i |\alpha_i|_E}{2k}$$

Since the union of the η -levers-the maximal η -levers and the two end η -levers, covers more than $(1 - \epsilon_1)|\gamma|_E$ part of γ^E , and we suppose that the two end η -levers have lengths less than l_1 ,

$$\geq (1 - \epsilon_1) \frac{(1 - \epsilon_1)|\gamma|_E - 2l_1}{2k}$$

Take $|\gamma|_E$ to be long enough, so that $|\gamma|_E \geq L_2 = \frac{2l_1}{\epsilon_1}$

$$\begin{aligned} &\geq (1 - \epsilon_1) \frac{(1 - 2\epsilon_1)|\gamma|_E}{2k} \\ &\geq \frac{(1 - 2\epsilon_1)^2|\gamma|_E}{2k} \end{aligned}$$

Hence, $(1 - \epsilon_1) \sum_i |\delta_i| \geq \frac{(1 - 2\epsilon_1)^2|\gamma|_E}{2k}$.

The ‘bad’ parts of γ^h are of three kinds. The first kind of bad part is the two end subsegments of γ^h which have lengths less than L_1 . The sum of the lengths of the end subsegments of γ^h is at most $2L_1$. We can take $|\gamma|_E$ to be big enough such that $2L_1 \leq \epsilon_1|\gamma|_E$.

The second kind of bad part of γ^h is the $\epsilon_1|\delta_i|$ part of δ_i ’s which may be out of the ϵ_1 neighborhood of σ_i . Since the projection map can not prolong length, and the distances between the ends of α_i and δ_i are not greater than N_0 ,

$$\sum_i \epsilon_1|\delta_i| \leq \epsilon_1 \sum_i (|\alpha_i| + 2N_0)$$

We can take η to be small enough, so that $|\alpha_i|_E \geq l_2 = 4kN_0$ for any i . The singular Euclidean metric and the hyperbolic metric are k bi-Lipschitz shows that $|\alpha_i|_E \leq k|\alpha_i|$. Therefore $2N_0 \leq 2kN_0 \leq \frac{|\alpha_i|}{2}$

$$\begin{aligned} &\leq \epsilon_1 \frac{3}{2} \sum_i |\alpha_i| \\ &\leq \epsilon_1 \frac{3}{2} k \sum_i |\alpha_i|_E \\ &\leq \epsilon_1 2k |\gamma|_E \end{aligned}$$

The third kind of bad part of γ^h are the projections of $\epsilon_1|\gamma|_E$ part of γ^E which has slope greater than ϵ_1 with f_ϕ^s . Let ξ_i denote this kind of subsegment of γ^E . There is a lower bound b of the Euclidean lengths of ξ_i for all i , which equals the minimum of the Euclidean distances between singularities. The sum of the lengths of the projections from ξ_i to γ^h is at most $\sum_i (k|\xi_i|_E + c) \leq \sum_i (k|\xi_i|_E + (n - 1)kb) \leq n \sum_i (k|\xi_i|_E) \leq nk\epsilon_1|\gamma|_E$, for some n satisfies $c \leq (n - 1)kb$.

Therefore, the length of the ‘bad’ part of γ^h is at most the sum of the above three kinds, which is $(2k + 1 + nk)\epsilon_1|\gamma|_E$. Hence the ratio of the ‘good’ part of γ^h to the ‘bad’ part of γ^h is at least $\frac{(1-2\epsilon_1)^2}{2k(1+2k+nk)\epsilon_1}$. It is easy to see, for any constant ϵ there exists a small enough ϵ_1 , such that the ratio of the ‘good’ part of γ^h to γ^h is at least $(1 - \epsilon)$.

To recap: for any $\epsilon > 0$, we can choose small enough ϵ_1 , so that $\frac{(1-2\epsilon_1)^2}{2k(1+2k+nk)\epsilon_1}$ is greater than $1 - \epsilon$, therefore the ‘good’ part of γ^h covers more than $(1 - \epsilon)$ of the total length of γ^h . Then choose η small enough so that if $\gamma^E \notin \text{slope}_\phi^\eta$, then more than $(1 - \epsilon)|\delta_i|$ part of δ_i has distance less than ϵ_1 with σ_i . In addition, take $|\gamma|_E$ to be at least L , where $L = \max\{L_2, 2L_1/\epsilon_1\}$. Hence if η is small enough, $\gamma \notin \text{slope}_\phi^\eta$ and $|\gamma|_E \geq L$, then most part of γ^h has distance at most ϵ to l_ϕ^s , measured in PS . \square

Given a geodesic lamination Λ and $0 < \epsilon < 1$, let $WN_\epsilon(\Lambda)$ denote the set of all the homotopy class γ , so that on a subset of γ^h of length at least $(1 - \epsilon)\text{Length}(\gamma^h)$, the distance from the tangent line of γ^h to the set Λ , measured in PS , is at most ϵ . Using this notation, the parallel corresponds lemma says that for any $0 < \epsilon < 1$, there exists $0 < \eta < 1$ and $L > 0$, such that if $\gamma \notin \text{slope}_\phi^\eta$ and $|\gamma|_E \geq L$, then $\gamma \in WN_\epsilon(\Lambda^s)$, where Λ^s is the measured stable geodesic lamination of ϕ .

3.2 Proof of the main theorem

Proof of Theorem 1. We shall prove that there exist $\lambda > 1$ and $C > 0$, so that for any vertex $w \in \Gamma$, if a based geodesic segment $\gamma_w^h \subset F_w$ has length at least C , then all but at most one preimages of it are stretched by corresponding $\phi_i^{m_i}$ by a factor of at least λ , for any $i \in I_w$, where $I_w = \{i | e_i \text{ is an oriented edge such that the origin of } e_i \text{ is } w\}$. Hence the hallways flare condition is satisfied. Therefore ST_{φ^m} is a hyperbolic surface.

Let v be a vertex of Γ , let $\gamma_v^h \subset F_v$ be a based geodesic segment. Consider the set $\Sigma = \bigcup_{i \in I_v} p_i^{-1}(\gamma_v^h)$, where $p_i^{-1}(\gamma_v^h)$ is the set of all preimages of γ_v^h under the map p_i . Notice that all the elements of Σ are based geodesics, since the edge surfaces of ST_{φ^m} equipped with the pullback metrics.

First, we claim that there exist $0 < \epsilon_0 < 1$ and $H_0 > 0$, such that if the length of γ_v^h is greater than H_0 , then at most one of the elements of Σ , say $\beta \in p_{i_0}^{-1}(\gamma_v^h)$, such that $\beta \in WN_{\epsilon_0}(\Lambda_{i_0}^s)$, for some $i_0 \in I_v$; all other elements of Σ are not contained in $WN_\epsilon(\Lambda_i^s)$ for corresponding Λ_i^s . Second, according to Lemma 5, for this ϵ_0 , there exist $0 < \eta(\epsilon_0) < 1$ and $L(\epsilon_0) > 0$, such that any $\alpha \in \Sigma$ with length $|\alpha| = |\gamma_v^h|$ greater than $L(\epsilon_0)$, if $\alpha \notin WN_{\epsilon_0}(\Lambda_j^s)$, then $\alpha \in \text{slope}_{\phi_j}^{\eta(\epsilon_0)}$. Therefore α is stretched by $\phi_j^{m_j}$ by a factor of at least λ for sufficiently large m_j . Combining these, we know that for any γ_v^h with length greater $C = \max\{H, L(\epsilon_0)\}$, all but at most one preimages of γ_v^h are stretched by corresponding $\phi_i^{m_i}$ by at least a factor λ .

Suppose the claim is not true. Namely for any $\epsilon_n \rightarrow 0$, and any $H_n \rightarrow \infty$, there exist based geodesic segments $\gamma_n^h \subset F_v$ with lengths at least H_n , by passing to a subsequence, without loss of generality, suppose $A_n^h \in p_1^{-1}(\gamma_n^h)$ and $B_n^h \in p_2^{-1}(\gamma_n^h)$,

such that $A_n^h \in WN_{\epsilon_n}(\Lambda_1^s)$ and $B_n^h \in WN_{\epsilon_n}(\Lambda_2^s)$. Project A_n^h and B_n^h to Λ_1^s and Λ_2^s respectively, there exist long subsegments $\nu_n \subset \Lambda_1^s$ and $\omega_n \subset \Lambda_2^s$, such that $|\nu_n|, |\omega_n| \rightarrow \infty$, and the distance between $Dp_1|T\nu_n$ and $Dp_2|T\omega_n$ converges to zero. This conflicts with the fact that $Dp_1|T\Lambda_1^s$ and $Dp_2|T\Lambda_2^s$ are disjoint. \square

3.3 Reformulation of Theorem 1

Notations here are the same as in the introduction. The only difference is the edge surfaces are not necessary equipped with the pullback metrics here.

Let v be a vertex of Γ , let y be the base point of F_v , and let I_v be as defined before. Consider the set $p_i^{-1}(y) \subset S_i$ of all the points of S_i that cover y via the map p_i , for $i \in I_v$. Denote $X = \cup_{i \in I_v} p_i^{-1}(y)$.

Suppose $a \in p_i^{-1}(y)$, choose a lift $\tilde{p}_a : (\tilde{S}_i, \tilde{a}) \rightarrow (\tilde{F}_v, \tilde{y})$, where \tilde{S}_i and \tilde{F}_v are the universal covers of S_i and F_v respectively. Let $\Lambda_i^s \subset S_i$ be the stable lamination of ϕ_i , and let $\tilde{\Lambda}_i^s \subset \tilde{S}_i$ be the lift of Λ_i^s . Notice that $\partial\tilde{p}_a(\tilde{\Lambda}_i^s) \subset \partial\tilde{F}_v$ is well defined independent of the choice of \tilde{y}, \tilde{a} . If for any $a \neq b \in X$, $\partial\tilde{p}_a(\tilde{\Lambda}_i^s) \cap \partial\tilde{p}_b(\tilde{\Lambda}_j^s) = \emptyset$, where $a \in p_i^{-1}(y)$, $b \in p_j^{-1}(y)$, then we say v satisfies the *disjointness condition*. We only ask $a \neq b$, but i may equal to j . The reformulation of Theorem 1 is the following.

Theorem 6. *Let ST_{φ^m} be a finite graph of surfaces with underlying graph Γ . If for any vertex $v \in \Gamma$, the disjointness condition is satisfied, then $\pi_1(ST_{\varphi^m})$ is a hyperbolic group, when $m_i \in \mathbf{m}$ are sufficiently large.*

We shall show the equivalence of the hypothesis of Theorem 1 and Theorem 6.

First, suppose $Dp_i(T\Lambda_i^s)$ is disjoint from $Dp_j(T\Lambda_j^s)$, for $i \neq j$. Then the images of the leaves Λ_i^s under the map p_i must transversely intersect the images of the leaves Λ_j^s under the map p_j . Thus the end points of their lifts in \tilde{F} are disjoint.

Second, suppose $Dp_i(T\Lambda_i^s)$ is injection for all i . If $\partial\tilde{p}_{a_1}(\tilde{\Lambda}_i^s) \cap \partial\tilde{p}_{a_2}(\tilde{\Lambda}_i^s) \neq \emptyset$, for some $a_1, a_2 \in p_i^{-1}(y)$, then there exist leaves $\tilde{L}_1, \tilde{L}_2 \subset \tilde{\Lambda}_i^s$, such that $\tilde{p}_{a_1}(\tilde{L}_1) = \tilde{p}_{a_2}(\tilde{L}_2)$. It contradicts with the injectiveness of $Dp_i(T\Lambda_i^s)$. We have finished the proof of one direction.

Suppose $Dp_i(T\Lambda_i^s)$ is not disjoint with $Dp_j(T\Lambda_j^s)$, i.e., there exist leaves $L \subset \Lambda_i^s$ and $J \subset \Lambda_j^s$, such that $Dp_i(L) = Dp_j(J)$. Therefore there exist a lift \tilde{L} of L , a lift \tilde{J} of J , such that $\tilde{p}_a(\tilde{L}) = \tilde{p}_b(\tilde{J})$ for some $a \in p_i^{-1}(y)$ and some $b \in p_j^{-1}(y)$. It conflicts with the hypothesis of Theorem 6. Similar proof for the injections of $Dp_i(T\Lambda_i^s)$ for all i .

4 Applications

The theorem below will be used to prove Corollary 8.

Theorem 7. (Farb & Mosher [FM02a], Theorem 1.2) Let $\pi_1(S)$ be the fundamental group of a surface S , and let Γ_α be the surface group extension of a group G . If Γ_α is word hyperbolic then the homomorphism $\alpha : G \rightarrow MCG$ has finite kernel and convex cocompact image.

Corollary 8. Let G, H be finite subgroups of $MCG(S)$, and let $\Phi \in MCG(S)$ be a pseudo-Anosov mapping class. If the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with G and H , then $\langle G, \Phi^M H \Phi^{-M} \rangle$ is a free product in $MCG(S)$, i.e., $\langle G, \Phi^M H \Phi^{-M} \rangle \cong G * \Phi^M H \Phi^{-M}$, and its extension group is hyperbolic, for sufficiently large M .

Remark: if G is a finite subgroup of $MCG(S)$, then G has a faithful representation, still called $G \subset Homeo(S)$. The quotient S/G , called F_0 , is a hyperbolic surface or orbifold. There exists a canonical embedding $i : \mathcal{PML}(F_0) \hookrightarrow \mathcal{PML}(S)$, where \mathcal{PML} is the projective measured geodesic laminations space. Given a pseudo-Anosov mapping class $\Phi \in MCG(S)$, if the stable and unstable geodesic laminations $\Lambda^s, \Lambda^u \notin i(\mathcal{PML}(F_0))$, then the virtual centralizer of $\langle \Phi \rangle$ has trivial intersection with G . Therefore, it is very easy to find $\Phi \in MCG(S)$ which satisfies the hypothesis of this corollary.

Proof. Let the symbols G, H denote both the finite groups of $MCG(S)$ and their faithful representations in $Homeo(S)$. Let $F_0 = S/G$, $F_1 = S/H$. Let $p : S \rightarrow F_0$, $q : S \rightarrow F_1$ denote the corresponding covering maps, and let $p_* : \pi_1(S) \rightarrow \pi_1(F_0)$, $q_* : \pi_1(S) \rightarrow \pi_1(F_1)$ denote the induced maps on fundamental groups.

Let $G\Gamma$ be the graph of groups:

$$\pi_1(F_0) \xleftarrow{p_*} \pi_1(S) \xrightarrow{\Phi^M} \pi_1(S) \xrightarrow{q_*} \pi_1(F_1)$$

$\pi_1(G\Gamma)$ is the fundamental group of the graph of surfaces $S\Gamma$:

$$F_0 \xleftarrow{p} S \xrightarrow{\phi^M} S \xrightarrow{q} F_1$$

where $\phi \in Homeo(S)$ is a pseudo-Anosov representative homeomorphism of Φ .

There exists a short exact sequence

$$1 \rightarrow \pi_1(S, x) \rightarrow \Gamma_{G * \Phi^M H \Phi^{-M}} \rightarrow G * \Phi^M H \Phi^{-M} \rightarrow 1$$

It is not hard to see that $\Gamma_{G * \Phi^M H \Phi^{-M}}$ is isomorphic to $\Gamma_G *_{\pi_1(S)} \Gamma_{\Phi^M H \Phi^{-M}}$, and $\Gamma_G *_{\pi_1(S)} \Gamma_{\Phi^M H \Phi^{-M}}$ is isomorphic to $\pi_1(G\Gamma)$.

According to Theorem 7, if $\pi_1(G\Gamma)$ is a word hyperbolic group, then $\delta : G * \Phi^M H \Phi^{-M} \rightarrow MCG(S)$ has finite kernel. Since G and $\Phi^M H \Phi^{-M}$ are finite groups, by applying Theorem 3.11 of Scott and Wall [SW79], a normal subgroup of $G * \Phi^M H \Phi^{-M}$ must be trivial or finite index. Therefore δ is an injection, which tells us that $\langle G, \Phi^M H \Phi^{-M} \rangle \cong G * \Phi^M H \Phi^{-M}$.

In order to prove $\pi_1(G\Gamma)$ is word hyperbolic, we only need to show that $S\Gamma$ is a hyperbolic graph of surfaces.

Let $y \in F_0$ be the base point, let $\{x_1, \dots, x_r\} = p^{-1}(y)$ denote the preimages of y under the covering map p , and let $\tilde{x}_i \in \tilde{S}$ be a covering point of x_i for $i \in \{1, \dots, r\}$. Let $\tilde{p}_i : (\tilde{S}, \tilde{x}_i) \rightarrow (F_0, y)$ be a lift of p , let $D_{ik} : (S, x_i) \rightarrow (S, x_k)$ be a deck transformation of covering map p , and let $\tilde{D}_{ik} : (\tilde{S}, \tilde{x}_i) \rightarrow (\tilde{S}, \tilde{x}_k)$ be a lift of D_{ik} .

According to Theorem 6, if $\partial\tilde{p}_i(\tilde{\Lambda}^s) \subset \partial\tilde{F}_0$ are pairwise disjoint on $\partial\tilde{F}_0$, and the similar condition holds on $\partial\tilde{F}_1$, then ST is hyperbolic.

In the following, we only prove that $\partial\tilde{p}_1(\tilde{\Lambda}^s)$ and $\partial\tilde{p}_2(\tilde{\Lambda}^s)$ are disjoint; a similar argument holds for the pairwise disjointness of $\{\partial\tilde{p}_i(\tilde{\Lambda}^s)\}$ for all $i \in \{1, \dots, r\}$, and the pairwise disjointness of $\{\partial\tilde{q}_j(\tilde{\Lambda}^u)\}$ for all j .

Since $\tilde{p}_1 = \tilde{p}_2\tilde{D}_{12}$, $\tilde{p}_1(\tilde{\Lambda}^s) = \tilde{p}_2\tilde{D}_{12}(\tilde{\Lambda}^s)$. Hence if the boundary points of the images of $\tilde{\Lambda}^s$ under \tilde{p}_1 and \tilde{p}_2 have one point in common, then $\tilde{D}_{12}(\tilde{\Lambda}^s)$ and $\tilde{\Lambda}^s$ have one end point in common. Since $\tilde{D}_{12}(\tilde{\Lambda}^s)$ and $\tilde{\Lambda}^s$ are the lift of the geodesic laminations $D_{12}(\Lambda^s)$ and Λ^s respectively, by Fact 3, we know $D_{12}(\Lambda^s) = \Lambda^s$, where D_{12} considered as an element of $G \subset MCG(S)$. Applying Theorem 3.5 in [Mos], if $D_{12}(\Lambda^s) = \Lambda^s$, then D_{12} is contained in the virtual centralizer of $\langle\Phi\rangle$. This contradicts with the hypothesis that the virtual centralizer of $\langle\Phi\rangle$ has trivial intersection with G . \square

Let \mathcal{G}_{ϕ^m} as in Figure 3, where S, F are genus 3 and 2 tori. Let $p : S \rightarrow F$ and $q : S \rightarrow F$ be covering maps, and let ϕ be a pseudo-Anosov homeomorphism of the mapping class Φ . Abusing of notations, we use D_p, D_q for both the deck transformations of p, q and the mapping classes of the deck transformations. It is easy to see that the deck transformation group GD_p of p contains only two elements, D_p and the identity, the same is true for the deck transformation group of q . Abusing of notations, we let GD_p denote both the deck transformation group of p and its image in $MCG(S)$.

Corollary 9. *Suppose $a : S^1 \rightarrow F$ and $c : S^1 \rightarrow S$ are simple closed curves such that $p^{-1}(a(S^1)) = c(S^1)$, $c(S^1) \subset q^{-1}(a(S^1))$, and $q^{-1}(a(S^1))$ is disconnected, as in Figure 4. In addition, suppose the virtual centralizer of $\langle\Phi\rangle$ has trivial intersection with the images of the deck transformation groups of p and q in $MCG(S)$. Then $\pi_1(\mathcal{G}_{\phi^m})$ is a hyperbolic group, when m is sufficiently large.*

Proof. Let z be the base point of F , let x_1, x_2 be the covering points of z through the covering map p , and let y_1, y_2 be the covering points of z through the covering map q . Let $\tilde{p}_1 : (\tilde{S}, \tilde{x}_1) \rightarrow (F, z)$ and $\tilde{p}_2 : (\tilde{S}, \tilde{x}_2) \rightarrow (F, z)$ be the lifts of p , and let $\tilde{D}_p : (\tilde{S}, \tilde{x}_1) \rightarrow (\tilde{S}, \tilde{x}_2)$ be the lift of D_p . Similar notations hold for q .

According to Theorem 6, we only need to show that $\{\partial\tilde{p}_1(\tilde{\Lambda}^s), \partial\tilde{p}_2(\tilde{\Lambda}^s), \partial\tilde{q}_1(\tilde{\Lambda}^u), \partial\tilde{q}_2(\tilde{\Lambda}^u)\}$ is a pairwise disjoint set.

First, we shall prove that $\partial\tilde{p}_1(\tilde{\Lambda}^s) \cap \partial\tilde{p}_2(\tilde{\Lambda}^s) = \emptyset$, $\partial\tilde{q}_1(\tilde{\Lambda}^u) \cap \partial\tilde{q}_2(\tilde{\Lambda}^u) = \emptyset$.

We know that $\tilde{p}_1(\tilde{\Lambda}^s) = \tilde{p}_2\tilde{D}_p(\tilde{\Lambda}^s)$. If $\partial\tilde{p}_1(\tilde{\Lambda}^s)$ and $\partial\tilde{p}_2(\tilde{\Lambda}^s)$ are not disjoint, then $\tilde{\Lambda}^s = \tilde{D}_p(\tilde{\Lambda}^s)$, as discussed in Corollary 8. It conflicts with the hypothesis that the virtual centralizer of $\langle\Phi\rangle$ has trivial intersection with GD_p and GD_q .

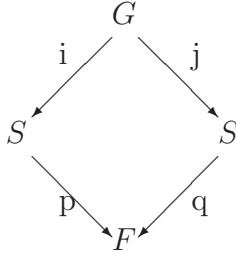
Therefore $\partial\tilde{p}_1(\tilde{\Lambda}^s)$ and $\partial\tilde{p}_2(\tilde{\Lambda}^s)$ are disjoint, the same holds for $\partial\tilde{q}_1(\tilde{\Lambda}^u)$ and $\partial\tilde{q}_2(\tilde{\Lambda}^u)$.

Second, we claim that if there exist $\partial\tilde{p}_r(\tilde{\Lambda}^s)$ and $\partial\tilde{q}_t(\tilde{\Lambda}^u)$ are not disjoint, for some $r, t \in \{1, 2\}$, then $p(\Lambda^s) = q(\Lambda^u)$ is a geodesic lamination on F . It follows that Λ^s is a fixed point of $GD_p \subset MCG(S)$. Therefore the virtual centralizer of $\langle \Phi \rangle$ and the deck transformation group have non-trivial intersection. A contradiction.

In the following, we will prove the above claim.

Since $p_*(\pi_1(S)) \neq q_*(\pi_1(S))$, and they are both index two subgroups of $\pi_1(F)$, $p_*(\pi_1(S)) \cap q_*(\pi_1(S))$ is an index 4 subgroup of $\pi_1(S)$. By calculating the Euler characteristic, we know there is a genus 5 surface G , and covering maps i and j , such that the diagram below commutes, i.e.,

$$pi = qj \tag{5}$$



After straightening, the preimages of $i^{-1}(\Lambda^s)$ and $j^{-1}(\Lambda^u)$ are geodesic laminations, called \mathcal{L}^s and \mathcal{L}^u , on G .

Without loss of generality, suppose $\tilde{p}_1(\tilde{\Lambda}^s)$ and $\tilde{q}_1(\tilde{\Lambda}^u)$ have one end point in common, then $\tilde{p}_1\tilde{i}(\tilde{\mathcal{L}}^s)$ and $\tilde{q}_1\tilde{j}(\tilde{\mathcal{L}}^u)$ have one end point in common. It follows that $\tilde{\mathcal{L}}^s$ and $\tilde{\mathcal{L}}^u$ have one common end point. We claim that \mathcal{L}^s and \mathcal{L}^u are minimal geodesic laminations and fill the surface G . Therefore if they have one common end point in the universal cover of G , then $\mathcal{L}^s = \mathcal{L}^u$. It is not hard to see that \mathcal{L}^s is connected and without isolated leaves, thereby \mathcal{L}^s is minimal according to Corollary 4.7.2 in [BC88]. \mathcal{L}^s and \mathcal{L}^u fill G because they are lifts of filling laminations Λ^s and Λ^u .

There exists some m , such that $\phi^m : S \rightarrow S$ is lifted by i and j to homeomorphisms of G respectively. Denote the lift of $\phi^m : S \rightarrow S$ through i as $\zeta : G \rightarrow G$, and the lift of ϕ^{-m} through j as $\sigma : G \rightarrow G$. Notice that \mathcal{L}^s is the stable geodesic lamination of ζ , \mathcal{L}^u is the stable geodesic lamination of σ . Since $\mathcal{L}^s = \mathcal{L}^u$, there exist positive integers k_1, k_2 , such that ζ^{k_1} is homotopic to σ^{k_2} .

Since ζ^{k_1} is homotopic to σ^{k_2} and $pi = qj$, we know: $pi\zeta^{k_1}$ is homotopic to $qj\sigma^{k_2}$. $p\phi^{k_1m}i$ is homotopic to $q\phi^{-k_2m}j$, because $\phi^{k_1m}i = i\zeta^{k_1}$ and $\phi^{-k_2m}j = j\sigma^{k_2}$.

$p(c)$ is a notation for the closed curve $p(c) : S^1 \rightarrow F$ which is the composition of $c : S^1 \rightarrow S$ with the covering map $p : S \rightarrow F$. Similar notations are used for other compositions of closed curves with covering maps. $c^2 : S^1 \rightarrow S$ is defined to be the composition of the map $z \rightarrow z^2$ on the unit circle S^1 with map $c : S^1 \rightarrow S$. Let $[a]$, $[c]$ denote the conjugacy classes in the fundamental group of F which represented by the simple closed curve a, c .

Since $p(c)$ is homotopic to a^2 and $q(c)$ is homotopic to a , it tells us that $[a] \notin p_*(\pi_1(S))$, $[a] \in q_*(\pi_1(S))$, and $[a]^2 \in p_*(\pi_1(S)) \cap q_*(\pi_1(S))$. Hence there exists $\gamma : S^1 \rightarrow G$ which is homotopic to a simple closed curve, such that $i(\gamma)$ is homotopic to c and $j(\gamma)$ is homotopic to c^2 . Therefore $p\phi^{k_1m}(c)$, $p\phi^{k_1m}i(\gamma)$, $q\phi^{-k_2m}j(\gamma)$ and $q\phi^{-k_2m}(c^2)$ are homotopic to each other.

We claim that $q\phi^{-k_2m}(c)$ is homotopic to a simple closed curve on F . Let β be the closed geodesic on F which is homotopic to $q\phi^{-k_2m}(c)$. If β is not simple, then there exists a point $z \in \beta(S^1)$, and a simple closed curve $\alpha : S^1 \rightarrow S$ which is homotopic to $\phi^{-k_2m}(c)$, such that $q(\alpha) = \beta$, and there exist two points $x_1 \neq x_2 \in \alpha(S^1)$ such that $q(x_1) = q(x_2) = z$. Since $p\phi^{k_1m}(c)$ is homotopic to $q\phi^{-k_2m}(c^2)$, there exists a simple closed curve η from S^1 to S which is homotopic to $\phi^{k_1m}(c)$, and whose image under the map p goes around β twice, to be more precise, $p(\eta) = \beta^2$. It follows that there are four different points $y_1, y_2, y_3, y_4 \in \eta(S^1)$ such that $p(y_1) = p(y_2) = p(y_3) = p(y_4) = z$, which conflicts with the fact that $p : S \rightarrow F$ is an index 2 covering map.

By iterating, we have:

$$\begin{aligned} pi\zeta^{nk_1} &\text{ is homotopic to } qj\sigma^{nk_2}, \text{ for all } n \in N \\ p\phi^{nk_1m}i(\gamma) &\text{ is homotopic to } q\phi^{-nk_2m}j(\gamma), \text{ for all } n \in N \\ p\phi^{nk_1m}(c) &\text{ is homotopic to } q\phi^{-nk_2m}(c^2), \text{ for all } n \in N \end{aligned}$$

By using the same argument, we know $q\phi^{-nk_2m}(c)$ is homotopic to a simple closed curve on F , for all $n \in N$. Let α_n denote the geodesics in the free homotopy class of $\phi^{-nk_2m}(c)$, there exists a subsequence of α_n , without loss of generality, still call it α_n , such that $\alpha_n \rightarrow \Lambda^u$ as $n \rightarrow \infty$. Since $q\phi^{-nk_2m}(c)$ is homotopic to a simple closed curve on F for all n , the geodesics in the free homotopic classes of $q\phi^{-nk_2m}(c)$ converge to a geodesic lamination $\Theta \subset F$, by passing to a subsequence. It follows that $q(\Lambda^u)$ is a geodesic lamination.

Notice that in the proof, we can only lift $\phi^m : S \rightarrow S$ by i and j to homeomorphisms of G for some $m \in N$, but the end points of $\partial\tilde{p}_i(\tilde{\Lambda}^s)$ and $\partial\tilde{q}_j(\tilde{\Lambda}^u)$ for any $i, j \in \{1, 2\}$ do not depend on m . Therefore we have proved that $\{\partial\tilde{p}_1(\tilde{\Lambda}^s), \partial\tilde{p}_2(\tilde{\Lambda}^s), \partial\tilde{q}_1(\tilde{\Lambda}^u), \partial\tilde{q}_2(\tilde{\Lambda}^u)\}$ is a pairwise disjoint set. According to Theorem 6, we know $\pi_1(\mathcal{G}_{\phi^m})$ is hyperbolic for sufficiently large m . \square

5 An example which is not abstractly commensurate to a surface-by-free group

In this section, we will show that there exist a graph of surfaces whose fundamental group is hyperbolic, but is not abstractly commensurate to any surface-by-free group, for any closed hyperbolic surface or orbifold S' and any free group K . Therefore this group is different from all the groups constructed in [Mos97]. By applying Theorem 1.1 in [FM02b], it follows that the example constructed here is not even quasi-isometric to any surface-by-free group.

Recall that, groups G and H are called *abstractly commensurate*, if there exist finite index subgroups $G_1 < G$ and $H_1 < H$, so that G_1 is isomorphic to H_1 . A group G is called a *surface-by-free* group, if there is a hyperbolic surface or a hyperbolic orbifold S , and a free group K , such that there exists a short exact sequence:

$$1 \rightarrow \pi_1(S) \rightarrow G \rightarrow K \rightarrow 1$$

First, we shall give a necessary and sufficient condition for a group to be abstractly commensurate to a surface-by-free group. Second, we shall construct a non-hyperbolic graph of surfaces \mathcal{G} , by applying the condition, whose fundamental group is not abstractly commensurate to any surface-by-free group. Finally, we shall construct a hyperbolic graph of surfaces \mathcal{G}_{ϕ^m} from \mathcal{G} such that $\pi_1(\mathcal{G}_{\phi^m})$ is not abstractly commensurate to any surface-by-free group.

Let t denote the Bass-Serre tree of a graph of surfaces ST , and let V, E denote the set of all the vertices and edges of t respectively. $\pi_1(ST)$ acts on t with subgroups $stab(v)$ and $stab(e)$, which stabilize the vertex $v \in V$ and the edge $e \in E$ respectively.

Lemma 10. *The fundamental group of a graph of surfaces ST is abstractly commensurate to a surface-by-free group if and only if $[stab(v) : \cap_{w \in V} stab(w)] < \infty$, for any $v \in V$.*

Proof. According to [FM02b], a finite index subgroup of a surface-by-free group is a surface-by-free group. If $\pi_1(ST)$ is abstractly commensurate to a surface-by-free group, then there exists a finite index subgroup H of $\pi_1(ST)$ which is isomorphic to a surface-by-free group.

H acts on t , and $[stab(v) : H \cap stab(v)] \leq [\pi_1(ST) : H]$ is finite. H acts on t with compact quotient, t may be identified with the Bass-Serre tree of H . Since H is isomorphic to a surface-by-free group $\pi_1(S') \rtimes F$, where S' is a hyperbolic surface, F is a finite rank free group, there exists a normal subgroup N of H which is isomorphic to $\pi_1(S')$, such that N acts trivially on t .

Let N denote $\cap_{w \in V} (stab(w) \cap H)$ which is a finite index subgroup of $stab(v) \cap H$ for any vertex $v \in t$, i.e., $[stab(v) \cap H : \cap_{w \in V} (stab(w) \cap H)] < \infty$. Therefore:

$$\begin{aligned} [stab(v) : \cap_{w \in V} stab(w)] &< [stab(v) : \cap_{w \in V} (stab(w) \cap H)] \\ &= [stab(v) : H \cap stab(v)][H \cap stab(v) : \cap_{w \in V} (stab(w) \cap H)] < \infty. \end{aligned}$$

We have finished the proof for one direction.

Now we will prove the other direction. The action of $\pi_1(ST)$ on t induces a homomorphism $\sigma : \pi_1(ST) \rightarrow Aut(t)$. Let $K = \cap_{w \in V} stab(w)$, $K = ker(\sigma)$. Since K is a finite index subgroup of $stab(v)$ for any $v \in V$, $\pi_1(ST)/K$ acts on t with finite edge and vertex stabilizers. In addition $\pi_1(ST)/K$ acts on t cocompactly. Therefore $t/(\pi_1(ST)/K)$ is a finite graph of finite groups. Applying Theorem 7.3 in [SW79], it follows that $\pi_1(ST)/K$ is virtually free. Hence $\pi_1(ST)$ is abstractly commensurate to a surface-by-free group. \square

In the rest of this paper, let \mathcal{G} denote a graph of surfaces as in Figure 5, where S, F, p, q and the simple closed curves $c \subset S, a \subset F$ are as described in Corollary 9.

The conclusion of the following lemma that $\pi_1(\mathcal{G})$ is not commensurate to a surface-by-free group was discovered and proved independently by Chris Odden in his thesis, and by Lee Mosher. I will give a different proof which will generalize to my later examples.

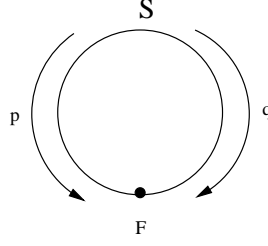


Figure 5:

Define subgroups L_i, R_i, G_i of $\pi_1(S)$, and subgroups H_i of $\pi_1(F)$ by induction as follows:

- let $H_0 = \pi_1(F)$, $G_0 = \pi_1(S)$,
- let $H_1 = p_*(G_0) \cap q_*(G_0)$, $L_1 = p_*^{-1}(H_1)$, $R_1 = q_*^{-1}(H_1)$, $G_1 = L_1 \cap R_1$,
- let $H_{i+1} = p_*(G_i) \cap q_*(G_i)$, $L_{i+1} = p_*^{-1}(H_{i+1})$, $R_{i+1} = q_*^{-1}(H_{i+1})$, $G_{i+1} = L_{i+1} \cap R_{i+1}$.

From $p_*([c]) = q_*([c^2]) = [a^2]$, we know $[a^2] \in H_1$ but $[a] \notin H_1$, $[c] \in L_1$, $[c^2] \in R_1$ but $[c] \notin R_1$.

Therefore $L_1 \neq R_1$, $[c^2] \in G_1$, $[c] \notin G_1$.

Similarly, from $p_*([c^2]) = [a^4]$, $q_*([c^2]) = [a^2]$, we know $p_*(G_1) \neq q_*(G_1)$, $[c^2] \in L_2$, $[c^4] \in R_2$, but $[c^2] \notin R_2$, $[c^4] \in G_2$.

Inductively, we have $[c^{2^n}] \in G_n$, $p_*([c^{2^n}]) = [a^{4^n}]$, $q_*([c^{2^n}]) = [a^{2^n}]$, so $p_*(G_n) \neq q_*(G_n)$, $[c^{2^n}] \in L_{n+1}$, $[c^{2^n}] \notin R_{n+1}$, but $[c^{2^{n+1}}] \in R_{n+1}$.

Hence we get two sequences $\{L_i\}$ and $\{R_i\}$ of finite index normal subgroups of $\pi_1(S)$, the indexes of $[\pi_1(S) : L_i]$ and $[\pi_1(S) : R_i] \rightarrow \infty$ as $i \rightarrow \infty$.

Lemma 11. *Suppose the edge group $\pi_1(S)$ of $\pi_1(\mathcal{G})$ contains two nested sequences of finite index normal subgroups $L_1 > L_2 > \dots$ and $R_1 > R_2 > \dots$ which are constructed inductively as follows:*

1. $H_1 = p_*(\pi_1(S)) \cap q_*(\pi_1(S))$, $L_1 = p_*^{-1}(H_1)$, $R_1 = q_*^{-1}(H_1)$, $G_1 = L_1 \cap R_1$
2. $H_{i+1} = p_*(G_i) \cap q_*(G_i)$
3. $L_{i+1} = p_*^{-1}(H_{i+1})$, $R_{i+1} = q_*^{-1}(H_{i+1})$, $G_{i+1} = L_{i+1} \cap R_{i+1}$

If $L_i \neq R_i$ for all i , then $\pi_1(\mathcal{G})$ is not commensurate to a surface-by-free group.

Proof. It is known that every edge or vertex stabilizer in the Bass-Serre tree t is isomorphic to some edge or vertex group of the graph of groups. Let e_1 be an edge of the Bass-Serre tree t such that the stabilizer $\text{stab}(e_1) = \pi_1(S)$. Let g be the generator of the underlying graph Γ of the graph of spaces \mathcal{G} ; [SW79] says that if $\pi_1(S)$ is identified with $p_*(\pi_1(S))$, then $q_*(\pi_1(S)) = g^{-1}\pi_1(S)g$. There exists a unique edge

$e_2 \in t$, such that $e_2 = ge_1$. It is easy to see that $R_1 = q_*^{-1}(p_*(\pi_1(S) \cap q_*(\pi_1(S))) = \text{stab}(e_1) \cap \text{stab}(e_2)$. Let $e_j = ge_{j-1}$ for a positive integer j , let α_i be the oriented path $e_1 * \dots * e_i$ in the Bass-Serre tree t . $\cap_{\epsilon \in \alpha_i} \text{stab}(\epsilon) = \cap_{j=1}^i \text{stab}(e_j) = R_i$. Similarly, there exists another sequence of oriented paths $\{\beta_k\}$ in t such that $\cap_{\epsilon \in \beta_k} \text{stab}(\epsilon) = L_k$. Therefore $[\pi_1(S) : L_i] \rightarrow \infty$ and $[\pi_1(S) : R_i] \rightarrow \infty$ imply $[\text{stab}(e) : \cap_{\epsilon \in E} \text{stab}(\epsilon)] = \infty$. For the case studied here, every edge stabilizer is a finite index subgroup of some vertex stabilizers, if the vertex is an end point of that edge. So $[\text{stab}(e) : \cap_{\epsilon \in E} \text{stab}(\epsilon)] = \infty$ implies $[\text{stab}(v) : \cap_{w \in V} \text{stab}(w)] = \infty$. According to Lemma 10, $\pi_1(\mathcal{G})$ is not commensurate to a surface-by-free group. \square

In order to construct a group which is not abstractly commensurate to a surface-by-free group, our first strategy is to find a pseudo-Anosov mapping class Φ which fixes all the finite index normal subgroups of $\pi_1(S)$. But unfortunately, the theorem below tells us that there does not exist such a pseudo-Anosov mapping class.

Theorem 12. *Let S_n be a closed surface with genus n , where $n \geq 2$. For any $\Phi \in \text{Aut}(\pi_1(S_n))$, if Φ fixes all the finite index normal subgroups of $\pi_1(S_n)$, then $\Phi \in \text{Inn}(\pi_1(S_n))$.*

Before proving this theorem, we introduce some related history and preliminaries first.

In [Lub80], Lubotzky proved that for any free group F_n , $n \geq 2$, if $\Psi \in \text{Aut}(F_n)$ fixes all the finite index normal subgroups of F_n , then $\Psi \in \text{Inn}(F_n)$. In particular, every normal automorphism of F_n is inner. Bogopolski, Kudryavtseva and Zieschang in [BKZ04] proved that for any closed hyperbolic surface S_n with genus n not less than 2, if $\Phi \in \text{Aut}(\pi_1(S_n))$ fixes all the normal subgroups of $\pi_1(S_n)$, then $\Phi \in \text{Inn}(\pi_1(S_n))$. The main theorem in that paper says for any non-separating simple closed curve α on S , up to conjugate equivalent, α is the only non-separating simple closed curve in its normal closure. The theorem in [BKZ04] says:

Theorem 13. *Let S be a closed orientable surface and g, h are non-trivial elements of $\pi_1(S)$ both containing simple closed two-sided curves γ and κ , resp. The group element h belongs to the normal closure of g if and only if h is conjugate to g^ϵ or to $(gug^{-1}u^{-1})^\epsilon$, $\epsilon \in \{1, -1\}$; here u is a homotopy class containing a simple closed curve μ which properly intersects γ exactly once.*

I would like to thank Jason DeBlois for help with Lemma 14.

Lemma 14. *For any two non trivial, non freely homotopic, non-separating simple closed curves a and b on S , let $[a], [b]$ denote the homotopy class of them in $\pi_1(S)$. There exists a finite index normal subgroup $N \in \pi_1(S)$, such that $[a] \in N$ and $[b] \notin N$.*

A group G is said to be *residually finite*, if for any element $g \in G$, $g \neq 1$, there exists a finite group K and a homomorphism $h : G \rightarrow K$, such that $h(g) \neq 1$.

A *Haken manifold* is a compact, orientable, irreducible 3-manifold which contains a 2-sided incompressible surface.

Proof. : Let $M = S \times I$, where I is the interval $[0, 1]$. $\pi_1(M)$ is isomorphic to $\pi_1(S)$. Since a is a simple closed curve on S , attach a 2-handle B to M along $a \times \{0\} \cup a \times \{1\}$ obtain a Haken manifold M' . This attachment gives a surjective homomorphism $\epsilon : \pi_1(M) \rightarrow \pi_1(M')$, and the kernel is the normal closure of $[a]$. Since a is the only non-separating simple closed curve in the normal closure of $[a]$, by applying Theorem 13, it follows that $[b]$ does not belong to the kernel of ϵ .

According to Theorem 1.1 in [Hem72], $\pi_1(M')$ is residually finite. So for $[b] \in \pi_1(M)$, there exist a finite group K and a homomorphism $\delta : \pi_1(M') \rightarrow K$, such that $[b] \notin \ker(\delta)$.

Let N denote the kernel $\ker(\delta \circ \epsilon)$. Obviously, N is a finite index normal subgroup of $\pi_1(S)$, and $[a] \in N$, but $[b] \notin N$. □

Proof of Theorem 12: Let Φ be an element of $\text{Aut}(\pi_1(S))$, and let ϕ be a representative of it in $\text{Homeo}(S)$. According to [BKZ04], if $\Phi \notin \text{Inn}(\pi_1(S))$, then there exists a non-separating simple closed curve a on S , such that a and $\phi(a)$ are not freely homotopic to each other. According to Lemma 14, there exist a finite index normal subgroup $N \triangleleft \pi_1(S)$, such that $[a] \in N$ and $[\phi(a)] \notin N$. It follows that $\Phi(N) \neq N$. □

In the following, we shall construct a pseudo-Anosov mapping class which does not fix all the finite index normal subgroups of $\pi_1(S)$, but fixes L_i and R_i as in Lemma 11.

In the following, let \mathcal{G}_{ϕ^m} be a graph of surfaces as in Figure 3, where F , S , p , q as described in Corollary 9.

Theorem 15. *There exists a pseudo-Anosov homeomorphism $\phi \in \text{Homeo}(S)$, so that $\pi_1(\mathcal{G}_{\phi^m})$ is hyperbolic but is not commensurate to a surface-by-free group.*

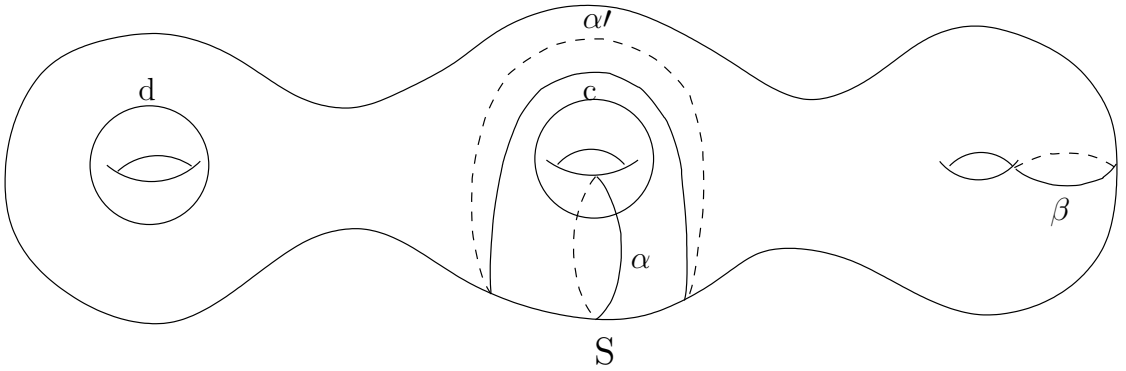


Figure 6:

Proof. If there exists a pseudo-Anosov homeomorphism ϕ , such that $\phi_*(L_i) = L_i$ and $\phi_*(R_i) = R_i$, according to Lemma 11, then $[\text{stab}(e) : \cap \text{stab}_{\epsilon \in E}(\epsilon)] = \infty$. Therefore $\pi_1(\mathcal{G}_{\phi^m})$ is not commensurate to a surface-by-free group.

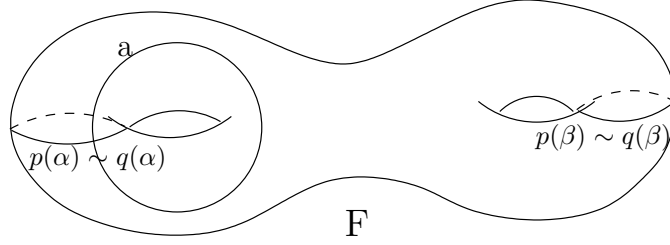


Figure 7:

Curves mentioned in this theorem are shown in Figure 6 and Figure 7, these two figures are a refinement of Figure 4.

First, we will describe the covering maps p and q with more details. Let $p^{-1}(a^2) = c$, $q^{-1}(a) = c \cup d$. It is easy to see that $p(\alpha)$ is homotopic to $q(\alpha) \subset F$ and $p(\beta)$ is homotopic to $q(\beta) \subset F$, where $\alpha, \beta \subset S$ as in Figure 6. Therefore $[\alpha], [\beta] \in L_i \cap R_i$ for all i .

Second, we claim that if γ is a simple closed curve in S , such that $[\gamma] \in L_i$ for some i , then $(\tau_\gamma)_*$ of the Dehn-twist τ_γ fixes L_i . Note that L_i is a finite index normal subgroup of $\pi_1(S)$ if and only if there exists a finite group K and a homomorphism $f : \pi_1(S) \rightarrow K$ such that $L_i = \ker(f)$. We shall see that τ_γ maps every element in the kernel of f to an element in the kernel of f , i.e., $(\tau)_*$ fixes L_i . Let $[g] \in \pi_1(S)$ be an element in L_i . Decompose $[g] = [h_1] \cdots [h_n]$, so that $[h_j] \in \pi_1(S)$ is represented by a closed loop in S which has only one transverse intersection point with γ , for all $j \in \{1, \dots, n\}$. Depending on how h_i intersects with γ , $[\tau_\gamma(h_i)]$ is one of the following four kinds: $[h_i\gamma]$, $[\gamma h_i]$, $[\gamma h_i \gamma^{-1}]$, $[\gamma^{-1} h_i \gamma]$. The trivial case is $g \cap \gamma = \emptyset$, so $f([\tau_\gamma(g)]) = [g]$. Otherwise,

$$\begin{aligned} f([\tau_\gamma(g)]) &= f([\tau_\gamma(h_1)] \cdots [\tau_\gamma(h_n)]) \\ &= f([\tau_\gamma(h_1)]) \cdots f([\tau_\gamma(h_n)]) \\ &= f([h_1]) \cdots f([h_n]) = f([h_1 \cdots h_n]) = f([g]) = I_k \end{aligned}$$

where I_k is the identity of K . It shows that $(\tau_\gamma)_*$ fixes L_i .

If we can find disjointly essential curve systems \mathcal{C} and \mathcal{D} which satisfy the conditions in Theorem 3, and if all the homotopy classes of the elements of \mathcal{C} and \mathcal{D} belong to L_i and R_i for all i , then we can construct a pseudo-Anosov homeomorphism ϕ as described in Theorem 3, such that ϕ_* fixes L_i and R_i for all i .

In the following, we will prove that there exist disjointly essential curve systems $\mathcal{C} = \alpha \cup \hat{\alpha}$, and $\mathcal{D} = \beta \cup \hat{\beta}$, such that $\mathcal{C} \cup \mathcal{D}$ fills S , where α, β as in Figure 6. In addition, $[\alpha], [\hat{\alpha}], [\beta]$ and $[\hat{\beta}] \in \cap_i (L_i \cap R_i)$.

In order to find a simple closed curve $\hat{\alpha}$ satisfying the above conditions, first, we will show that there exists a simple closed curve α' such that $[\alpha'] \in \cap_i (L_i \cap R_i)$. Since L_i and R_i are finite index normal subgroups of $\pi_1(S)$, and $[\alpha] \in \cap_i (L_i \cap R_i)$, the

normal closure N_α of $[\alpha]$ is a subgroup of $\cap_i (L_i \cap R_i)$. Recall that the normal closure N_α of $[\alpha]$ is the smallest normal subgroup of $\pi_1(S)$ contains $[\alpha]$. Applying Theorem 13, we only need the easy direction of this theorem, the separating curve α' as in Figure 6 represents an element in N_α .

Second, we shall construct a simple closed curve $\hat{\alpha}$ on S from the simple closed curve α' .

From [Mos03], we know there exists a short exact sequence:

$$1 \rightarrow \langle T_\alpha \rangle \rightarrow \text{stab}(\alpha) \rightarrow MCG(S - \alpha) \rightarrow 1$$

where $\langle T_\alpha \rangle$ is a cyclic subgroup of $MCG(S)$ generated by the mapping class T_α of the Dehn-twist τ_α around α , $\text{stab}(\alpha)$ is a subgroup of $MCG(S)$ which fixes α , $S - \alpha$ is a surface by cutting S along α . The homomorphism $\iota : \text{stab}(\alpha) \rightarrow MCG(S - \alpha)$ is defined as $\Phi \rightarrow \Phi|_{S-\alpha}$, for $\Phi \in \text{stab}(\alpha)$.

Choose a pseudo-Anosov homeomorphism $\psi \in \text{Homoeo}(S - \alpha)$, maybe need pass to a high enough power of ψ , such that $\hat{\alpha} = \psi(\alpha')$ is very close to the stable geodesic lamination Λ_ψ^S of ψ , therefore $\hat{\alpha} \cup \beta$ fills $S - \alpha$. Also $\hat{\alpha}$ is disjoint with α because α' is disjoint with α .

Using the same method, we can find a simple closed curve $\hat{\beta}$ which is disjoint with β and $\hat{\beta} \cup \alpha$ fills $S - \beta$.

Let $\mathcal{C} = \{\alpha, \hat{\alpha}\}$, $\mathcal{D} = \{\beta, \hat{\beta}\}$, it is easy to see that $\mathcal{C} \cup \mathcal{D}$ fills S . According to Theorem 3, if ϕ_0 is a homeomorphism of S , such that τ_α^+ , τ_α^- , τ_β^+ and τ_β^- appear at least once in ϕ_0 , then ϕ_0 is a pseudo-Anosov homeomorphism. Since $[\alpha], [\widehat{\alpha}], [\beta], [\widehat{\beta}] \in \cap_i (L_i \cap R_i)$, $(\phi_0)_*$ fixes L_i and R_i for all i .

In order to finish the proof of this theorem, according to Corollary 9, we only need to show that there exists some pseudo-Anosov homeomorphism ϕ constructed as above, so that the virtual centralizer $VC\langle\Phi\rangle$ of $\langle\Phi\rangle$ has trivial intersection with the mapping classes of the deck transformation groups of the covering maps p and q respectively, where $\Phi \in MCG(S)$ is the mapping class of ϕ . Abusing of notation, denote both the deck transformations and the mapping classes of the deck transformations by D_p and D_q . The deck transformation group of p has only two elements D_p and the identity.

Let ϕ_0 be a pseudo-Anosov homeomorphism of S constructed above, and let Φ_0 be its mapping class. Let $\Lambda_{\phi_0}^s$ and $\Lambda_{\phi_0}^u$ be the stable and unstable geodesic laminations of ϕ_0 respectively. It is known that Φ_0 fixes L_i and R_i for all i .

Suppose the deck transformation group of p has nontrivial intersection with the virtual centralizer of $\langle\Phi_0\rangle$, i.e., $D_p(\Lambda_{\phi_0}^s) = \Lambda_{\phi_0}^s$. We claim that $D_p(T_\alpha(\Lambda_{\phi_0}^s)) \neq T_\alpha(\Lambda_{\phi_0}^s)$, where T_α is the mapping class of the Dehn-twist τ_α . Notice that $T_\alpha(\Lambda_{\phi_0}^s)$ is the stable geodesic lamination of the pseudo-Anosov mapping class $T_\alpha\Phi_0T_\alpha^{-1}$, and $T_\alpha\Phi_0T_\alpha^{-1}$ fixes L_i and R_i for all i . If the claim is true, let $\Phi_1 = T_\alpha\Phi_0T_\alpha^{-1}$, then $VC\langle\Phi_1\rangle$ has trivial intersection with D_p .

We shall prove the claim. Notice that there exists a simple closed curve γ on S is disjoint with α , such that $D_p(\alpha) = \gamma$. According to Lemma 4.1.C in [Iva02], $D_pT_\alpha D_p^{-1} = T_{D_p(\alpha)} = T_\gamma$.

Suppose $D_p T_\alpha(\Lambda_{\phi_0}^s) = T_\alpha(\Lambda_{\phi_0}^s)$, then:

$$\begin{aligned} D_p T_\alpha(\Lambda_{\phi_0}^s) &= D_p T_\alpha \Phi_0 T_\alpha^{-1}(T_\alpha(\Lambda_{\phi_0}^s)) \\ &= T_\gamma D_p \Phi_0 T_\alpha^{-1}(T_\alpha(\Lambda_{\phi_0}^s)) \\ &= T_\gamma D_p \Phi_0(\Lambda_{\phi_0}^s) = T_\gamma(\Lambda_{\phi_0}^s) \end{aligned}$$

Therefore $T_\alpha(\Lambda_{\phi_0}^s) = T_\gamma(\Lambda_{\phi_0}^s)$.

It follows that $T_\alpha^{-1} T_\gamma \in VC\langle\Phi_0\rangle$, but from Theorem 3.5 in [Mos], we know that $VC\langle\Phi_0\rangle$ has $\langle\Phi_0\rangle$ as a finite index subgroup. Hence up to some power m , $(T_\alpha^{-1} T_\gamma)^m \in \langle\Phi_0\rangle$, but obviously $(T_\alpha^{-1} T_\gamma)^m$ is neither pseudo-Anosov nor the identity, so it is not an element of $\langle\Phi_0\rangle$. Therefore $D_p T_\alpha(\Lambda_{\phi_0}^s) \neq T_\alpha(\Lambda_{\phi_0}^s)$.

If in addition $D_q T_\alpha(\Lambda_{\phi_0}^u) \neq T_\alpha(\Lambda_{\phi_0}^u)$, then take $\Phi = \Phi_1$, this theorem is proved.

If $D_q T_\alpha(\Lambda_{\phi_0}^u) = T_\alpha(\Lambda_{\phi_0}^u)$, then we claim $D_q T_\alpha^2(\Lambda_{\phi_0}^u) \neq T_\alpha^2(\Lambda_{\phi_0}^u)$. If the claim is not true, then

$$\begin{aligned} D_q T_\alpha^2(\Lambda_{\phi_0}^u) &= T_\alpha^2(\Lambda_{\phi_0}^u) \\ &= T_\alpha(D_q T_\alpha(\Lambda_{\phi_0}^u)) \\ &= D_q T_\theta(T_\alpha(\Lambda_{\phi_0}^u)), \end{aligned}$$

where $\theta = D_q(\alpha)$ is a simple closed curve on S disjoint from α . Therefore

$$T_\alpha^{-1} T_\theta^{-1} D_q^{-1} D_q T_\alpha^2(\Lambda_{\phi_0}^u) = \Lambda_{\phi_0}^u$$

It follows that $T_\alpha^{-1} T_\theta^{-1} T_\alpha^2(\Lambda_{\phi_0}^u) = \Lambda_{\phi_0}^u$. Since θ, α are disjoint simple closed curves, $T_\alpha T_\theta^{-1} = T_\theta^{-1} T_\alpha$. Hence $T_\alpha^{-1} T_\theta^{-1} T_\alpha^2(\Lambda_{\phi_0}^u) = T_\theta^{-1} T_\alpha(\Lambda_{\phi_0}^u) = \Lambda_{\phi_0}^u$. By the same reason in the above argument, it is impossible.

Replacing T_α, T_γ by T_α^2, T_γ^2 in the above proof of $D_p T_\alpha(\Lambda_{\phi_0}^s) \neq T_\alpha(\Lambda_{\phi_0}^s)$, we can see $D_p T_\alpha^2(\Lambda_{\phi_0}^s) \neq T_\alpha^2(\Lambda_{\phi_0}^s)$. Take $\Phi = T_\alpha^2 \Phi_0 T_\alpha^{-2}$, then this theorem is proved. \square

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